International Journal of Uncertainty,
Fuzziness and Knowledge-Based Systems
Vol. 21, No. 3 (2013) 367–390

DOI: 10.1142/S0218488513500190

INTUITIONISTIC FUZZY ORDERED INFORMATION SYSTEM

WEIHUA XU* and YUFENG LIU†

School of Mathematics and Statistics, Chongqing University of Technology, Chongqing, 400054, P. R. China
**chxuwh@gmail.com*
†liuyufeng@cqut.edu.cn

TONGJUN LI

School of Mathematics, Physics and Information Science, Zhejiang Ocean University, Zhoushan, Zhejiang, 316000, P. R. China
litj@zjou.edu.cn

Received 26 July 2010
Revised 30 October 2012

In this paper, we aim to study intuitionistic fuzzy ordered information systems. Firstly, the concept of intuitionistic fuzzy ordered information systems is proposed by introducing an intuitionistic fuzzy relation to ordered information systems. Meanwhile, two approximation operators are defined and a rough set approach is established in intuitionistic fuzzy ordered information systems. Secondly, a ranking approach for all objects is constructed in this system. Thirdly, approximation reduction is addressed in intuitionistic fuzzy ordered decision information system. These results will be helpful for decision-making analysis in intuitionistic fuzzy ordered information systems.

Keywords: Rough set; ordered information system; intuitionistic fuzzy relation; attribute reduction; discernibility matrix.

1. Introduction

Partition or equivalence (indiscernibility relation) is an important concept in Pawlak's rough set theory, which was initiated by Pawlak in the early 1980s,[24] and is a new mathematical approach to uncertain date analysis. However, partition or equivalence relation is still restrictive for many applications. To overcome this limitation, classical rough sets have been extended to several interesting and meaningful general models in recent years by proposing other binary relations, such as tolerance relations,[33] neighborhood operators,[44] and others.[20,25,26,35,37,38,40,46] However, the original rough set theory does not consider attributes with preference

*Corresponding author.

ordered domain, that is criteria. Particularly, in many real situations, we are often face to the problems in which the ordering of properties of the considered attributes plays a crucial role. One such type of problem is the ordering of objects. For this reason, Greco, Matarazzo, and Slowinski[13–18] proposed an extension rough set theory, called the dominance-based rough set approach(DRSA) to take into account the ordering properties of criteria. This innovation is mainly based on substitution of the indiscernibility relation by a dominance relation. In DRSA, condition attributes are criteria and classes are preference ordered, the knowledge approximated is a collection of upward and downward unions of classes and the dominance classed are sets of objects defined by using a dominance relation. In recent years, several studies have been made about properties and algorithmic implementations of DRSA.[9,10,32,34,41,48]

Another important mathematical structure to cope with imperfect and/or imprecise information is called the "intuitionistic fuzzy (IF, for short) set"[a] initiated by Atanassov[1,2] on the basis of ortho-pairs of fuzzy sets. An IF set is naturally considered as an extension of Zadeh's fuzzy sets[47] defined by a pair of membership functions: while a fuzzy set gives a degree to which an element belongs to a set, an IF set gives both a membership degree and a non-membership degree. The membership and non-membership values induce an indeterminacy index, which models the hesitancy of deciding the degree to which an object satisfies a particular property. Recently, IF set theory has been successfully applied in decision analysis and pattern recognition.[11,22,36,43]

Combining IF set theory and rough set theory may result in a new hybrid mathematical structure for the requirement of knowledge-handling systems. Research on this topic has been investigated by a number of authors. Coker[7] first revealed the relationship between IF set theory and rough set theory and showed that a fuzzy rough set was in fact an intuitionistic fuzzy set. Various tentative definitions of IF rough sets were explored to extend rough set theory to the IF environment.[6,8,19,29–31,49] For example, according to fuzzy rough sets in the sense of Nanda and Majumda,[23] Jena and Ghosh[19] and Chakrabarty *et al.*[6] independently proposed the concept of an IF rough set in which the lower and upper approximations are both IF sets.

In this paper, the intuitionistic fuzzy information system is introduced to DRSA. Actually, in real life, the intuitionistic fuzzy information system is an important type of data tables in ordered information systems. We aim to introduce dominance relation to intuitionistic fuzzy information system, and establish a rough set approach and evidence theory in this system.

The rest of this paper is organized as follows. The intuitionistic fuzzy ordered information system is introduced, and some important properties are discussed in Sec. 2. In Sec. 3, a rough set approach is investigated by establishing the upper

[a]Though the term of intuitionistic fuzzy set has been the argument of a large debate,[3,5,12] we still use this notion due to its underlying mathematical structure, and because it is becoming increasing popular topic of investigation in the fuzzy set community.

and lower approximation operators in the system. In Sec. 4, a rank approach with dominance class is considered by proposing the concept of dominance degree intuitionistic fuzzy ordered information system. In Sec. 5, approximation reduction are proposed for the intuitionistic fuzzy ordered decision information system. Moreover, the judgement theorems and discernibility matrices associated with two reductions are obtained. Finally, we conclude the paper with a summary.

2. Intuitionistic Fuzzy Ordered Information Systems

Dominance-based rough set approach (DRSA) was proposed by Greco, Matarazzo, and Slowinski,[13–18] which is mainly based on a dominance relation. In this section, we introduce a dominance relation to the intuitionistic fuzzy information systems. We propose a new extension of information systems referred to as intuitionistic fuzzy ordered information systems. Moreover, we introduce a dominance relation to the new information systems, and obtain some of its important properties.

Firstly, we review a special lattice on $[0,1] \times [0,1]$ and some basic definitions of Intuitionistic fuzzy sets which will be used in this paper.

Definition 2.1.[4] Let $L^* = \{(\alpha_1, \alpha_2) \in [0,1] \times [0,1] | \alpha_1 + \alpha_2 \leq 1\}$. We define a relation $\leq_{L^*}$ on L^* as follows:

$$\begin{aligned}
&\forall (\alpha_1, \alpha_2), (\beta_1, \beta_2) \in L^*, \\
&(\alpha_1, \alpha_2) \leq_{L^*} (\beta_1, \beta_2) \Leftrightarrow \alpha_1 \leq \beta_1 \text{ and } \alpha_2 \geq \beta_2, \\
&(\alpha_1, \alpha_2) \not\leq_{L^*} (\beta_1, \beta_2) \Leftrightarrow \alpha_1 > \beta_1 \text{ or } \alpha_2 < \beta_2.
\end{aligned}$$

Then the relation $\leq_{L^*}$ is a partial ordering on L^* and the pair $(L^*, \leq_{L^*})$ is a complete lattice with the smallest element $0_{L^*} = (0,1)$ and the greatest element $1_{L^*} = (1,0)$. The meet operator $\wedge$, join operator $\vee$ and complement operator $\sim$ on $(L^*, \leq_{L^*})$ which are linked to the ordering $\leq_{L^*}$ are, respectively, defined as follows:

$$\begin{aligned}
&\forall (\alpha_1, \alpha_2), (\beta_1, \beta_2) \in L^*, \\
&(\alpha_1, \alpha_2) \wedge (\beta_1, \beta_2) = (\min(\alpha_1, \beta_1), \max(\alpha_2, \beta_2)), \\
&(\alpha_1, \alpha_2) \vee (\beta_1, \beta_2) = (\max(\alpha_1, \beta_1), \min(\alpha_2, \beta_2)), \\
&\sim (\alpha_1, \alpha_2) = (\alpha_2, \alpha_1).
\end{aligned}$$

Definition 2.2.[1] Let a set U be fixed. An IF set $\widetilde{X}$ in U is an object having the form

$$\widetilde{X} = \{\langle x, \mu_{\widetilde{X}}(x), \nu_{\widetilde{X}}(x)\rangle | x \in U\},$$

where $\mu_{\widetilde{X}} : U \to I$ and $\nu_{\widetilde{X}} : U \to I$ satisfy $0 \leq \mu_{\widetilde{X}}(x) + \nu_{\widetilde{X}}(x) \leq 1$ for all $x \in U$, $\mu_{\widetilde{X}}(x)$ and $\nu_{\widetilde{X}}(x)$ are called the degree of membership and the degree of non-membership of the element $x \in U$ to $\widetilde{X}$, respectively. The family of all IF subsets of U is denoted by $IF(U)$. The complement of an IF set $\widetilde{X}$ is defined by $\sim \widetilde{X} = \{\langle x, \nu_{\widetilde{X}}(x), \mu_{\widetilde{X}}(x)\rangle | x \in U\}$.

To discuss conveniently the problem of our paper, some basic operations between any two intuitionistic fuzzy sets $\widetilde{X}, \widetilde{Y} \in IF(U)$ can be defined as follows.

$$\widetilde{X} \subseteq \widetilde{Y} \Leftrightarrow \mu_{\widetilde{X}}(x) \leq \mu_{\widetilde{Y}}(x) \text{ and } \nu_{\widetilde{X}}(x) \geq \nu_{\widetilde{Y}}(x) \text{ for any } x \in U,$$
$$\widetilde{X} = \widetilde{Y} \Leftrightarrow \mu_{\widetilde{X}}(x) = \mu_{\widetilde{Y}}(x) \text{ and } \nu_{\widetilde{X}}(x) = \nu_{\widetilde{Y}}(x) \text{ for any } x \in U.$$

An intuitionistic fuzzy information system is an ordered quadruple $\mathcal{I}^{\wr} = (U, AT, V, f)$, where $U = \{x_1, x_2, \cdots, x_n\}$ is a non-empty finite set of objects called universe, and $AT = \{a_1, a_2, \cdots, a_p\}$ is a non-empty finite set of attributes, $V = \bigcup_{a \in AT} V_a$ and V_a is a domain of attribute a, $f : U \times AT \longrightarrow V$ is a function such that $f(x, a) \in V_a$, for all $a \in AT, x \in U$, called an information function, where V_a is an intuitionistic fuzzy set of U. That is

$$f(x, a) = (\mu_a(x), \nu_a(x)), \text{for all } a \in AT\,,$$

where $\mu_a : U \to [0, 1]$ and $\nu_a : U \to [0, 1]$ satisfy $0 \leq \mu_a(x) + \nu_a(x) \leq 1$, for all $x \in U$. And $\mu_a(x)$ and $\nu_a(x)$ are, respectively, called the degree of membership and the degree of non-membership of the element $x \in U$ to attribute a. We denote $\tilde{a}(x) = (\mu_a(x), \nu_a(x))$, then it is clear that $\tilde{a}$ is an intuitionistic fuzzy set of U.

An intuitionistic fuzzy decision information system is a special intuitionistic fuzzy information system $\mathcal{I}^{\wr} = (U, AT \cup \{d\}, V, f)$, where A is called conditional attribute set, and d is an decision attribute of $\mathcal{I}^{\wr}$. Thus that is to say $\tilde{d}(x) = f(x, d) = (\mu_d(x), \nu_d(x))$ for all $u \in U$ in $\mathcal{I}^{\wr}$.

Example 2.1. An intuitionistic fuzzy decision information system is presented in Table 1, where $U = \{x_1, x_2, x_3, x_4, x_5, x_6\}$, $AT = \{a_1, a_2, a_3\}$.

Table 1. An intuitionistic fuzzy decision information system.

U	a_1	a_2	a_3	d
x_1	$(0.2, 0.7)$	$(0.6, 0.4)$	$(0.5, 0.2)$	3
x_2	$(0.8, 0.1)$	$(0.6, 0.4)$	$(0.6, 0.4)$	2
x_3	$(0.2, 0.7)$	$(0.1, 0.8)$	$(0.6, 0.4)$	1
x_4	$(0.3, 0.5)$	$(0.1, 0.8)$	$(0.8, 0.1)$	2
x_5	$(0.8, 0.1)$	$(0.8, 0.1)$	$(0.6, 0.4)$	3
x_6	$(0.8, 0.1)$	$(0.6, 0.4)$	$(0.8, 0.1)$	1

In practical decision making analysis, we always consider a binary dominance relation between objects that are possibly dominant in terms of values of attributes set in intuitionistic fuzzy information systems. In general, an increasing preference and a decreasing preference, then the attribute is a criterion.

Definition 2.3. An intuitionistic fuzzy information system is called an intuitionistic fuzzy ordered information system (IFOIS) if all condition attributes are criteria.

In general, we denote an intuitionistic fuzzy ordered information system by $\mathcal{I}^{\wr\succeq} = (U, AT, V, f)$.

Assumed that the domain of a criterion $a \in AT$ is complete pre-ordered by an outranking relation $\succeq_a$, then $x \succeq_a y$ means that x is at least as good as y with respect to criterion a. And we can say that x dominates y. For a subset of attributes $A \subseteq AT$, we define $x \succeq_A y \Longleftrightarrow \forall a \in A, x \succeq_a y$. In other words, x is at least as good as y with respect to all attributes in A.

In the following, we introduce a dominance relation to an intuitionistic fuzzy information system. In a given IFOIS, we say that x dominates y with respect to $A \subseteq AT$ if $x \succeq_A y$, and denoted by $xR_A^{\wr\succeq}y$. That is

$$R_A^{\wr\succeq} = \{(x, y) \in U \times U | x \succeq_A y\}.$$

Obviously, if $(x, y) \in R_A^{\wr\succeq}$, then x dominates y with respect to A. $R_A^{\wr\succeq}$ are called a dominant relations of IFOIS.

Similarly, the relation $R_A^{\wr\preceq}$, which is called a dominated relation, can be defined as following

$$R_A^{\wr\preceq} = \{(x, y) \in U \times U | y \succeq_A x\}.$$

For simplicity and without any loss of generality, in the following we only consider condition attributes with increasing preference. Let us define this dominant relation in intuitionistic fuzzy ordered information systems as follows,

$$R_A^{\wr\succeq} = \{(x, y) \in U \times U | \mu_a(x) \geq \mu_a(y) \text{ and } \nu_a(x) \leq \nu_a(y), \forall a \in A\}.$$

That is to say that $R_A^{\wr\succeq}$ is called dominance relation of IFOIS $\mathcal{I}^{\wr\succeq}$.

Let denote

$$\begin{aligned}
[x_i]_A^{\wr\succeq} &= \{x_j \in U | (x_j, x_i) \in R_A^{\wr\succeq}\} \\
&= \{x_j \in U | \mu_a(x_j) \geq \mu_a(x_i) \text{ and } \nu_a(x_j) \leq \nu_a(x_i), \forall a \in A\}; \\
U/R_A^{\wr\succeq} &= \{[x_i]_A^{\wr\succeq} | x_i \in U\},
\end{aligned}$$

where $i \in \{1, 2, \cdots, |U|\}$, then $[x_i]_A^{\wr\succeq}$ describes the set of objects that may dominate x_i in terms of A in IFOIS $\mathcal{I}^{\wr\succeq}$, and will be called a dominance class of IFOIS $\mathcal{I}^{\wr\succeq}$, and $U/R_A^{\wr\succeq}$ be called a classification of U about attribute set A in IFOIS $\mathcal{I}^{\wr\succeq}$.

An intuitionistic fuzzy ordered decision information system is an intuitionistic fuzzy ordered information system $\mathcal{I}^{\wr\succeq} = (U, AT \cup \{d\}, V, f)$, where the relation induced by d is equivalence relation.

Definition 2.4. Let $\mathcal{I}^{\wr\succeq} = (U, AT, V, f)$ be an intuitionistic fuzzy ordered information system and $B, A \subseteq AT$.

(1) If $[x]^{\succsim}_B = [x]^{\succsim}_A$ for any $x \in U$, then we call that classification $U/R^{\succsim}_B$ is equal to $U/R^{\succsim}_A$, denoted by $U/R^{\succsim}_B = U/R^{\succsim}_A$.

(2) If $[x]^{\succsim}_B \subseteq [x]^{\succsim}_A$ for any $x \in U$, then we call that classification $U/R^{\succsim}_B$ is finer than $U/R^{\succsim}_A$, denoted by $U/R^{\succsim}_B \subseteq U/R^{\succsim}_A$.

(3) If $[x]^{\succsim}_B \subseteq [x]^{\succsim}_A$ for any $x \in U$ and $[x]^{\succsim}_B \neq [x]^{\succsim}_A$ for some $x \in U$, then we call that classification $U/R^{\succsim}_B$ is properly finer then $U/R^{\succsim}_A$, denoted by $U/R^{\succsim}_B \subset U/R^{\succsim}_A$.

From the definition of $R^{\wr\succeq}_A$ and $[x]^{\wr\succeq}_A$, we can obtain some properties as follows.

Proposition 2.1. *Let $\mathcal{I}^{\wr\succeq} = (U, AT, V, f)$ be an intuitionistic fuzzy ordered information system, and $A \subseteq AT$, we can have*

$$R^{\wr\succeq}_A = \bigcap_{a \in A} R^{\wr\succeq}_{\{a\}}\,.$$

Proposition 2.2. *Let $\mathcal{I}^{\wr\succeq} = (U, AT, V, f)$ be an intuitionistic fuzzy ordered information system, and $A \subseteq AT$. Then*

(1) $R^{\wr\succeq}_A$ *is reflexive,*

(2) $R^{\wr\succeq}_A$ *is unsymmetric, and*

(3) $R^{\wr\succeq}_A$ *is transitive.*

Proposition 2.3. *Let $\mathcal{I}^{\wr\succeq} = (U, AT, V, f)$ be an intuitionistic fuzzy ordered information system, and $A, B \subseteq AT$, we have the following results.*

(1) *If $B \subseteq A$, then $R^{\wr\succeq}_A \subseteq R^{\wr\succeq}_B$.*

(2) *If $B \subseteq A$, then $[x_i]^{\wr\succeq}_A \subseteq [x_i]^{\wr\succeq}_B$*

(3) *If $x_j \in [x_i]^{\wr\succeq}_A$, then $[x_j]^{\wr\succeq}_A \subseteq [x_i]^{\wr\succeq}_A$ and $[x_i]^{\wr\succeq}_A = \cup\{[x_j]^{\wr\succeq}_A | x_j \in [x_i]^{\wr\succeq}_A\}$.*

(4) *$[x_j]^{\wr\succeq}_A = [x_i]^{\wr\succeq}_A$ iff $\mu_a(x_i) = \mu_a(x_j)$ and $\nu_a(x_i) = \nu_a(x_j)$ for all $a \in A$.*

These properties mentioned above can be understood through the following example.

Example 2.2. (Continued from Example 2.1) Computing the classification induced by the dominance relation $R^{\wr\succeq}_{AT}$ in Table 1.

From the table, One can obtain that

$$\begin{aligned}
[x_1]^{\succcurlyeq}_{AT} &= \{x_1, x_2, x_5, x_6\};\\
[x_2]^{\succcurlyeq}_{AT} &= \{x_2, x_5, x_6\};\\
[x_3]^{\succcurlyeq}_{AT} &= \{x_2, x_3, x_4, x_5, x_6\};\\
[x_4]^{\succcurlyeq}_{AT} &= \{x_4, x_6\};\\
[x_5]^{\succcurlyeq}_{AT} &= \{x_5\};\\
[x_6]^{\succcurlyeq}_{AT} &= \{x_6\}\,.
\end{aligned}$$

If take $A = \{a_1, a_3\} \subseteq AT$, we can get that

$$
\begin{aligned}
[x_1]_A^{\succeq} &= \{x_1, x_2, x_3, x_4, x_5, x_6\};\\
[x_2]_A^{\succeq} &= \{x_2, x_5, x_6\};\\
[x_3]_A^{\succeq} &= \{x_2, x_3, x_4, x_5, x_6\};\\
[x_4]_A^{\succeq} &= \{x_4, x_6\};\\
[x_5]_A^{\succeq} &= \{x_2, x_5, x_6\};\\
[x_6]_A^{\succeq} &= \{x_6\}.
\end{aligned}
$$

Obviously, $[x_i]_{AT}^{\wr\succeq} \subseteq [x_i]_A^{\wr\succeq}$.

According to this example, we can easily verify above propositions of intuitionistic fuzzy ordered information systems.

3. Rough Set Approach to IFOIS

In this section, we investigate the problem of set approximation with respect to a dominance relation $R_A^{\wr\succeq}$ in intuitionistic fuzzy ordered information systems.

Definition 3.1. Let $\mathcal{I}^{\wr\succeq} = (U, AT, V, f)$ be an intuitionistic fuzzy ordered information system. For any $X \subseteq U$ and $A \subseteq AT$, the upper and lower approximations of X with respect to the dominance relation $R_A^{\wr\succeq}$ are defined as follows:

$$
\begin{aligned}
\overline{R_A^{\wr\succeq}}(X) &= \{x \in U | [x]_A^{\wr\succeq} \cap X \neq \emptyset\};\\
\underline{R_A^{\wr\succeq}}(X) &= \{x \in U | [x]_A^{\wr\succeq} \subseteq X\}.
\end{aligned}
$$

From above definition, one can briefly notice that $\underline{R_A^{\wr\succeq}}(X)$ is a set of objects that belong to X with certainty and $\overline{R_A^{\wr\succeq}}(X)$ is a set of objects that probably belong to X. If $\underline{R_A^{\wr\succeq}}(X) \neq \overline{R_A^{\wr\succeq}}(X)$, we say the subset X of U is rough, otherwise X is precise. $Bn_A(X) = \overline{R_A^{\wr\succeq}}(X) - \underline{R_A^{\wr\succeq}}(X)$ is called to a boundary of the rough set.

Moreover, we can directly obtain the following results.

Proposition 3.1. *Let $\mathcal{I}^{\wr\succeq} = (U, AT, V, f)$ be an intuitionistic fuzzy ordered information system and $A \subseteq AT$. For any $X \subseteq U$, the following always hold.*

(1) $\underline{R_A^{\wr\succeq}}(X) \subseteq \underline{R_{AT}^{\wr\succeq}}(X)$ *and* $\overline{R_A^{\wr\succeq}}(X) \supseteq \overline{R_{AT}^{\wr\succeq}}(X)$.

(2) *If* $R_A^{\wr\succeq} = R_{AT}^{\wr\succeq}$, *then* $\underline{R_A^{\wr\succeq}}(X) = \underline{R_{AT}^{\wr\succeq}}(X)$ *and* $\overline{R_A^{\wr\succeq}}(X) = \overline{R_{AT}^{\wr\succeq}}(X)$.

Proposition 3.2. *Let $\mathcal{I}^{\wr\succeq} = (U, AT, V, f)$ be an intuitionistic fuzzy ordered* *information system. For any* $X, Y \subseteq U$ *and* $A \subseteq AT$*, then*

(1L) $\underline{R_A^{\wr\succeq}}(X) \subseteq X$ (Contraction)

(1U) $X \subseteq \overline{R_A^{\wr\succeq}}(X)$ (Extension)

(2) $\underline{R_A^{\wr\succeq}}(\sim X) = \sim \overline{R_A^{\wr\succeq}}(X)$ (Duality)

$\overline{R_A^{\wr\succeq}}(\sim X) = \sim \underline{R_A^{\wr\succeq}}(X)$ (Duality)

(3L) $\underline{R_A^{\wr\succeq}}(\emptyset) = \emptyset$ (Normality)

(3U) $\overline{R_A^{\wr\succeq}}(\emptyset) = \emptyset$ (Normality)

(4L) $\underline{R_A^{\wr\succeq}}(U) = U$ (Co-normality)

(4U) $\overline{R_A^{\wr\succeq}}(U) = U$ (Co-normality)

(5L) $\underline{R_A^{\wr\succeq}}(X \cap Y) = \underline{R_A^{\wr\succeq}}(X) \cap \underline{R_A^{\wr\succeq}}(Y)$ (Multiplication)

(5U) $\overline{R_A^{\wr\succeq}}(X \cup Y) = \overline{R_A^{\wr\succeq}}(X) \cup \overline{R_A^{\wr\succeq}}(Y)$ (Addition)

(5L') $\underline{R_A^{\wr\succeq}}(X \cup Y) \supset \underline{R_A^{\wr\succeq}}(X) \cup \underline{R_A^{\wr\succeq}}(Y)$ (F-Multiplication)

(5U') $\overline{R_A^{\wr\succeq}}(X \cap Y) \subset \overline{R_A^{\wr\succeq}}(X) \cap \overline{R_A^{\wr\succeq}}(Y)$ (F-Addition)

(6L) $X \subseteq Y \Rightarrow \underline{R_A^{\wr\succeq}}(X) \subseteq \underline{R_A^{\wr\succeq}}(Y)$ (Monotone)

(6U) $X \subseteq Y \Rightarrow \overline{R_A^{\wr\succeq}}(X) \subseteq \overline{R_A^{\wr\succeq}}(Y)$ (Monotone)

(7L) $\underline{R_A^{\wr\succeq}}(\underline{R_A^{\wr\succeq}}(X)) = \underline{R_A^{\wr\succeq}}(X)$ (Idempotency)

(7U) $\overline{R_A^{\wr\succeq}}(\overline{R_A^{\wr\succeq}}(X)) = \overline{R_A^{\wr\succeq}}(X)$ (Idempotency)

Proof. The proof are similar to the case of properties in Ref. 31. □

Example 3.1. Consider the intuitionistic fuzzy ordered information system in Table 1.

According to Examples 2.1 and 2.2, computing the upper and lower approximation of decision attribute $\widetilde{d}$.

One can obtain that $\mathbf{d} = \{\mathbf{D_1}, \mathbf{D_2}, \mathbf{D_3}\}$, where

$$D_1 = [x_1]_d^{\succcurlyeq} = [x_5]_d^{\succcurlyeq} = \{x_1, x_5\}\,,$$
$$D_2 = [x_2]_d^{\succcurlyeq} = [x_4]_d^{\succcurlyeq} = \{x_2, x_4\}\,,$$
$$D_3 = [x_3]_d^{\succcurlyeq} = [x_6]_d^{\succcurlyeq} = \{x_3, x_6\}\,.$$

Then, we have

$$\underline{R_A^{\succcurlyeq}}(D_1) = \{x_5\}, \qquad \overline{R_A^{\succcurlyeq}}(D_1) = \{x_1, x_2, x_3, x_5\};$$
$$\underline{R_A^{\succcurlyeq}}(D_2) = \emptyset, \qquad \overline{R_A^{\succcurlyeq}}(D_2) = \{x_1, x_2, x_3, x_4, x_5\};$$
$$\underline{R_A^{\succcurlyeq}}(D_3) = \{x_6\}, \qquad \overline{R_A^{\succcurlyeq}}(D_3) = \{x_1, x_2, x_3, x_4, x_5, x_6\}\,.$$

4. Ranking for Objects in IFOIS

In general, there are two classes of problems in intelligent decision-making. One is to find satisfactory results through ranking with information aggregation. And the other is to find dominance rules through relations.

In this section, we mainly investigate that how to rank all objects by the dominance relation in an ordered information system based on intuitionistic fuzzy relation.

Definition 4.1. Let $\mathcal{I}^{\wr\succeq} = (U, AT, V, f)$ be an ordered information system based on intuitionistic fuzzy relation and $A \subseteq AT$. Dominance degree between two objects $x_i, x_j \in U$ with respect to the dominance relation $R_A^{\wr\succeq}$ is defined as

$$d_A(x_i, x_j) = 1 - \frac{|[x_i]_A^{\wr\succeq} \cap (\sim [x_j]_A^{\wr\succeq})|}{|U|}\,.$$

We say that dominance degree of x_i to x_j is $d_A(x_i, x_j)$.

From the definition, the dominance degree $d_A(x_i, x_j)$ depict the proportion of some objects which are as least as good as x_j in dominance class $[x_i]_A^{\wr\succeq}$. Moreover, we can obtain the following properties.

Proposition 4.1. *Let $\mathcal{I}^{\wr\succeq} = (U, AT, V, f)$ be an ordered information system based on intuitionistic fuzzy relation, $A \subseteq AT$ and dominance degree between two objects x_j and x_i be $d_A(x_i, x_j)$ with respect to the dominance relation $R_A^{\wr\succeq}$, then the following hold.*

(1) $0 \le d_A(x_i, x_j) \le 1$ and $d_A(x_i, x_i) = 1$.

(2) *If $x_i \in [x_j]_A^{\wr\succeq}$, then $d_A(x_i, x_j) = 1$.*

(3) *If $x_j \in [x_k]_A^{\wr\succeq}$, then $d_A(x_i, x_j) \le d_A(x_i, x_k)$.*

(4) *If $x_j \in [x_k]_A^{\wr\succeq}$ and $x_k \in [x_i]_A^{\wr\succeq}$, then $d_A(x_i, x_j) \le d_A(x_k, x_j)$ and $d_A(x_i, x_j) \le d_A(x_i, x_k)$.*

Proof.

(1) is directly obtained by the definition.

(2) Since $x_i \in [x_j]_A^{\wr\succeq}$, one can have $[x_i]_A^{\wr\succeq} \subseteq [x_j]_A^{\wr\succeq}$ by Proposition 3.3. So, we have $[x_i]_A^{\wr\succeq} \cap (\sim [x_j]_A^{\wr\succeq}) = \emptyset$. That is to say

$$d_A(x_i, x_j) = 1 - \frac{|[x_i]_A^{\wr\succeq} \cap (\sim [x_j]_A^{\wr\succeq})|}{|U|} = 1\,.$$

(3) If $x_j \in [x_k]_A^{\wr\succeq}$, then we can obtain $[x_j]_A^{\wr\succeq} \subseteq [x_k]_A^{\wr\succeq}$. So we have $(\sim [x_j]_A^{\wr\succeq}) \supseteq (\sim [x_k]_A^{\wr\succeq})$. Thus

$$\frac{|[x_i]_A^{\wr\succeq} \cap (\sim [x_j]_A^{\wr\succeq})|}{|U|} \ge \frac{|[x_i]_A^{\wr\succeq} \cap (\sim [x_k]_A^{\wr\succeq})|}{|U|}\,.$$

Thus

$$d_A(x_i, x_j) \leq d_A(x_i, x_k)\,.$$

(4) If $x_j \in [x_k]_A^{\wr\succeq}$ and $x_k \in [x_i]_A^{\wr\succeq}$, then we can obtain $[x_j]_A^{\wr\succeq} \subseteq [x_k]_A^{\wr\succeq} \subseteq [x_i]_A^{\wr\succeq}$. That is $(\sim [x_j]_A^{\wr\succeq}) \supseteq (\sim [x_k]_A^{\wr\succeq}) \supseteq (\sim [x_i]_A^{\wr\succeq})$ hold. So we have

$$\frac{|[x_i]_A^{\wr\succeq} \cap (\sim [x_j]_A^{\wr\succeq})|}{|U|} \geq \frac{|[x_k]_A^{\wr\succeq} \cap (\sim [x_j]_A^{\wr\succeq})|}{|U|}\,,$$

and

$$\frac{|[x_i]_A^{\wr\succeq} \cap (\sim [x_j]_A^{\wr\succeq})|}{|U|} \geq \frac{|[x_i]_A^{\wr\succeq} \cap (\sim [x_k]_A^{\wr\succeq})|}{|U|}\,.$$

Thus

$$d_A(x_i, x_j) \leq d_A(x_k, x_j), \quad d_A(x_i, x_j) \leq d_A(x_i, x_k)\,.$$

The proposition was proved. □

Definition 4.2. Let $\mathcal{I}^{\wr\succeq} = (U, AT, V, f)$ be an ordered information system based on intuitionistic fuzzy relation and $A \subseteq AT$. Denote

$$M_A^{\wr\succeq} = (r_{ij})_{|U|\times|U|}, \text{ where } \; r_{ij} = d_A(x_i, x_j)\,.$$

Then, we call the matrix $M_A^{\wr\succeq}$ to be a dominance matrix with respect to A induced by the intuitionistic fuzzy dominance relation $R_A^{\wr\succeq}$.

Moreover, if denote

$$d_A(x_i) = \frac{1}{|U|} \sum_{x_j \in U} d_A(x_i, x_j)\,,$$

then we call $d_A(x_i)$ to be dominance degree of x_i with respect to relation $R_A^{\wr\succeq}$, for every $x_i \in U$.

By definition of dominance matrix and dominance degree of the object with respect to relation $R_A^{\wr\succeq}$, we can directly receive the following properties. For all $x_i \in U$, the degree can be calculated according to the following formula

$$d_A(x_i) = \frac{1}{|U|} \sum_{j=1}^{|U|} r_{ij}\,.$$

As a result of the above discussions, we come to the following two corollaries.

Corollary 4.1. *Let $\mathcal{I}^{\wr\succeq} = (U, AT, V, f)$ be an ordered information system based on intuitionistic fuzzy relation and $A \subseteq AT$. If $R_A^{\wr\succeq} = R_{AT}^{\wr\succeq}$, then $d_A(x_i, x_j) = d_{AT}(x_i, x_j)$, $d_A(x_i) = d_{AT}(x_i)$ and $M_A^{\wr\succeq} = M_{AT}^{\wr\succeq}$, for $x_i, x_j \in U$.*

From the dominance degree of each object on the universe, we can rank all objects according to the number of d_A. A larger number implies a better object. This idea can be understood by the following example.

Example 4.1. (Continued From Example 3.1.) Rank all objects in U according to the dominance relation $R^{\wr\succeq}_{AT}$ in the system of Example 3.1.

By Example 3.2, we can easily obtain the dominance degree of two objects and dominance matrix in the system as follows:

$$M^{\wr\succeq}_{AT} = \begin{bmatrix} 1 & \frac{5}{6} & \frac{5}{6} & \frac{3}{6} & \frac{3}{6} & \frac{3}{6} \\ 1 & 1 & 1 & \frac{4}{6} & \frac{4}{6} & \frac{4}{6} \\ \frac{4}{6} & \frac{4}{6} & 1 & \frac{3}{6} & \frac{2}{6} & \frac{2}{6} \\ \frac{5}{6} & \frac{5}{6} & 1 & 1 & \frac{4}{6} & \frac{5}{6} \\ 1 & 1 & 1 & 1 & 1 & \frac{5}{6} \\ 1 & 1 & 1 & 1 & \frac{5}{6} & 1 \end{bmatrix}.$$

So, we can have

$$d_{AT}(x_1) = 0.69,\ \ d_{AT}(x_2) = 0.83,\ \ d_{AT}(x_3) = 0.58,$$
$$d_{AT}(x_4) = 0.86,\ \ d_{AT}(x_5) = 0.97,\ \ d_{AT}(x_6) = 0.97.$$

Therefore, according the above we rank all objects in the following.

$$x_5 = x_6 \succeq x_4 \succeq x_2 \succeq x_1 \succeq x_3\,.$$

5. Approximation Reduction and Rules Extracted from IFODIS

The approximation reduction proposed by Mi *et al.* is an important attribute reduction, which can be used to simplify an inconsistent classical decision table [14], and extract more briefer rules. So far, however, there is not any practical approach to attribute reduction in intuitionistic fuzzy ordered decision information system. In this section, we present the notions of lower approximation reduction and upper approximation reductions in intuitionistic fuzzy ordered decision information system, and then we develop the method based on discernibility matrix to compute all approximation approximation reductions. Moreover, we investigate rules extracted from intuitionistic fuzzy ordered decision information system.

Definition 5.1. Let $\mathcal{I}^{\wr\succeq} = (U, AT \cup \{d\}, V, f)$ be an IFODIS and $B \subseteq AT$.

If $\overline{R^{\wr\succeq}_{B}}(D_i) = \overline{R^{\wr\succeq}_{AT}}(D_i)$ for any $D_i \in U/R^{\wr\succeq}_{d}$, then we say that B is upper approximation consistent set of this information system. Moreover, if any proper subset of B is not the upper approximation consistent set, then B is called to one upper approximation reduction of this information system.

If $\underline{R^{\wr\succeq}_{B}}(D_i) = \underline{R^{\wr\succeq}_{AT}}(D_i)$ for any $D_i \in U/R^{\wr\succeq}_{d}$, then we say that B is lower approximation consistent set of this information system. Moreover, if any proper subset of B is not a lower approximation consistent set, then B is called to one lower approximation reduction of this information system.

From above definition, one can find that upper and lower approximation consistent sets preserve the upper and lower approximations of the fuzzy decision respectively.

Detailed judgment theorems of upper and lower approximation reductions will be proposed in the following two propositions.

Proposition 5.1. *Let $\mathcal{I}^{\wr\succeq} = (U, AT \cup \{d\}, V, f)$ be an IFODIS and $B \subseteq AT$. Attribute set B is an upper approximation consistent set iff $x \notin \overline{R^{\succcurlyeq}_{AT}}(D_i)$, $y \in \overline{R^{\succcurlyeq}_{AT}}(D_i)$ for some $D_i \in U/R^{\succcurlyeq}_{d}$, then there must exist $a \in B$ such that $f(x,a) \nleq_{L^*} f(y,a)$.*

Proof. "$\Longrightarrow$" Suppose that the conclusion does not holds, that is to say that if exist D_i such that $x \notin \overline{R^{\succcurlyeq}_{AT}}(D_i)$, $y \in \overline{R^{\succcurlyeq}_{AT}}(D_i)$, then $f(x,a) \leq_{L^*} f(y,a)$ for any $a \in B$. So we can obtain $y \in [x]^{\wr\succcurlyeq}_{B}$, which implies that $[y]^{\wr\succcurlyeq}_{B} \subseteq [x]^{\wr\succcurlyeq}_{B}$. And, by the upper approximation definition, we have known that $\overline{R^{\wr\succcurlyeq}_{B}}(D_i) = \overline{R^{\wr\succcurlyeq}_{AT}}(D_i)$ for any $D_i \in U/R^{\wr\succeq}_{d}$, so $y \in \overline{R^{\wr\succcurlyeq}_{AT}}(D_i)$, i.e., $[y]^{\wr\succcurlyeq}_{B} \bigcap D_i \neq \emptyset$. Therefore, one can get $[x]^{\wr\succcurlyeq}_{B} \bigcap D_i \neq \emptyset$, we have $x \in \overline{R^{\wr\succcurlyeq}_{B}}(D_i) = \overline{R^{\wr\succcurlyeq}_{AT}}(D_i)$. Obviously, this is a contradiction.

"$\Longleftarrow$" Suppose that B is not an upper approximation consistent set, then there exist certainly one $D_i \in U/R^{\succcurlyeq}_{d}$ such that $\overline{R^{\wr\succcurlyeq}_{B}}(D_i) \neq \overline{R^{\wr\succcurlyeq}_{AT}}(D_i)$, i.e., exist $x_0 \in U$ such that $x_0 \notin \overline{R^{\wr\succcurlyeq}_{AT}}(D_i)$ and $x_0 \in \overline{R^{\wr\succcurlyeq}_{B}}(D_i)$. Then we have $[x_0]^{\wr\succcurlyeq}_{AT} \bigcap D_i = \emptyset$ and $[x_0]^{\wr\succcurlyeq}_{B} \bigcap D_i \neq \emptyset$. Then exist $y_0 \in [x_0]^{\wr\succcurlyeq}_{B} \bigcap D_i$, we have $y_0 \in [y_0]^{\wr\succcurlyeq}_{AT} \bigcap D_i$, therefor $y_0 \in \overline{R^{\wr\succcurlyeq}_{AT}}(D_i)$. Thus there must exist $a \in B$ such that $f(x,a) \nleq_{L^*} f(y,a)$, then $y_0 \notin [x_0]^{\wr\succcurlyeq}_{B}$. Obviously, this is a contradiction.

The proposition is proved. $\square$

Proposition 5.2. *Let $\mathcal{I}^{\wr\succeq} = (U, AT \cup \{d\}, V, f)$ be an IFODIS and $B \subseteq AT$. Attribute set B is an lower approximation consistent set iff $x \in \overline{R^{\succcurlyeq}_{AT}}(D_i)$, $y \notin \overline{R^{\succcurlyeq}_{AT}}(D_i)$ for some $D_i \in U/R^{\succcurlyeq}_{d}$, then there must exist $a \in B$ such that $f(x,a) \nleq_{L^*} f(y,a)$.*

Proof. It is similar to Proposition 5.1.

Propositions 5.1 and 5.2 provide an approach to judge whether a subset of condition attributes is a lower and upper approximation consistent set or not, respectively.

Definition 5.2. Let $\mathcal{I}^{\wr\succeq} = (U, AT \cup \{d\}, V, f)$ be an IFOIS and $B \subseteq AT$. If we denote, for $x_i, x_j \in U$,

$$UD^*_{\tilde{d}} = \{(x_i, x_j) \mid x_i \notin \overline{R^{\succcurlyeq}_{AT}}(D_i), x_j \in \overline{R^{\succcurlyeq}_{AT}}(D_i)\}, \quad \exists D_i \in U/R^{\succcurlyeq}_{d}$$

$$LD^*_{\tilde{d}} = \{(x_i, x_j) \mid x_i \in \underline{R^{\succcurlyeq}_{AT}}(D_i), x_j \notin \underline{R^{\succcurlyeq}_{AT}}(D_i)\}, \quad \exists D_i \in U/R^{\succcurlyeq}_{d}$$

$$UD_{\tilde{d}}(x_i, x_j) = \begin{cases} \{a \in C | f(x_i, a) \nleq_{L^*} f(x_j, a)\}, & (x_i, x_j) \in UD^*_{\tilde{d}} \\ \varnothing, & (x_i, x_j) \notin UD^*_{\tilde{d}} \end{cases},$$

$$LD_{\tilde{d}}(x_i, x_j) = \begin{cases} \{a \in C | f(x_i, a) \nleq_{L^*} f(x_j, a)\}, & (x_i, x_j) \in LD^*_{\tilde{d}} \\ \varnothing, & (x_i, x_j) \notin LD^*_{\tilde{d}} \end{cases},$$

and

$$UM_{\tilde{d}} = (u_{ij})_{n\times n},$$

$$LM_{\tilde{d}} = (v_{ij})_{n\times n},$$

where $u_{ij} = UD_{\tilde{d}}(x_i, x_j)$, $v_{ij} = LD_{\tilde{d}}(x_i, x_j)$, then $UD_{\tilde{d}}(x_i, x_j)$ and $LD_{\tilde{d}}(x_i, x_j)$ are said to be upper and lower approximation discernibility attributes set between objects x_i and x_j, respectively. And matrices $UM_{\tilde{d}}$ and $LM_{\tilde{d}}$ are referred as to upper and lower approximation discernibility matrix of the system $\mathcal{I}^{\wr\succeq}$ respectively.

Proposition 5.3. *Let $\mathcal{I}^{\wr\succeq} = (U, AT \cup \{d\}, V, f)$ be an IFODIS and $B \subseteq AT$. Subset B is upper approximation consistent set if and only if $B \cap UD_{\tilde{d}}(x_i, x_j) \neq \varnothing$ for all $(x_i, x_j) \in UD^*_{\tilde{d}}$.*

Proof. "$\Longrightarrow$" From $(x_i, x_j) \in UD^*_{\tilde{d}}$, then exist $D_i \in U/R^{\succeq}_{\tilde{d}}$ such that $x \notin \overline{R^{\succeq}_{AT}}(D_i)$, $y \in \overline{R^{\succeq}_{AT}}(D_i)$. By proposition 5.1, we can know that there exist certainly $a \in B$ such that $f(x_i, a) \nleq_{L^*} f(x_j, a)$. So $a \in UD_{\tilde{d}}(x_i, x_j)$ according to the above definition. Hence, $B \cap UD_{\tilde{d}}(x_i, x_j)(x_i, x_j) \neq \varnothing$.

"$\Longleftarrow$" For all $(x_i, x_j) \in UD^*_{\tilde{d}}$, i.e., $x_i \notin \overline{R^{\succeq}_{AT}}(D_i)$, $x_j \in \overline{R^{\succeq}_{AT}}(D_i)$, if $B \cap UD_{\tilde{d}}(x_i, x_j) \neq \varnothing$, then there exist certainly $a \in B$ such that $a \in UD_{\tilde{d}}(x_i, x_j)$, which implies that $f(x_i, a) \nleq_{L^*} f(x_j, a)$. By Proposition 5.1, we can obtain that B is an upper approximation consistent set of the system $\mathcal{I}^{\wr}$. □

Proposition 5.4. *Let $\mathcal{I}^{\wr\succeq} = (U, AT \cup \{d\}, V, f)$ be an IFODIS and $B \subseteq AT$. Subset B is lower approximation consistent set if and only if $B \cap LD_{\tilde{d}}(x_i, x_j) \neq \varnothing$ for all $(x_i, x_j) \in LD^*_{\tilde{d}}$.*

Proof. It is similar to Proposition 5.3. □

Definition 5.3. Let $\mathcal{I}^{\wr\succeq} = (U, AT \cup \{d\}, V, f)$ be an IFODIS. $UM_{\tilde{d}}$ and $LM_{\tilde{d}}$ be upper and lower approximation discernibility matrices of $\mathcal{I}^{\wr}$ respectively. If denote

$$\begin{aligned} UF_{\tilde{d}} &= \wedge\{\vee\{a \mid a \in UD_{\tilde{d}}(x_i, x_j)\}, x_i, x_j \in U\} \\ &= \wedge\{\vee\{a \mid a \in UD_{\tilde{d}}(x_i, x_j)\}, (x_i, x_j) \in UD^*_{\tilde{d}}\}, \\ LF_{\tilde{d}} &= \wedge\{\vee\{a \mid a \in LD_{\tilde{d}}(x_i, x_j)\}, x_i, x_j \in U\} \\ &= \wedge\{\vee\{a \mid a \in LD_{\tilde{d}}(x_i, x_j)\}, (x_i, x_j) \in LD^*_{\tilde{d}}\}; \end{aligned}$$

then $UF_{\tilde{d}}$ and $LF_{\tilde{d}}$ are called discernibility formulas of upper and lower approximation of the system $\mathcal{I}^{\wr\succeq}$ respectively.

Proposition 5.5. *Let $\mathcal{I}^{\wr\succeq} = (U, AT \cup \{d\}, V, f)$ be an IFODIS. The minimal disjunctive normal form of discernibility formula of upper approximation is*

$$UF_{\tilde{d}} = \bigvee_{k=1}^{p} \left(\bigwedge_{s=1}^{q_k} \right) a_s \,.$$

Denote $UB_{\tilde{d}}^{k} = \{a_s \mid s = 1, 2, \cdots, q_k\}$, then $\{UB_{\tilde{d}}^{k} \mid k = 1, 2, \cdots, p\}$ is just set of all upper approximation reductions of $\mathcal{I}^{\wr\succeq}$.

Proof. For any $(x_i, x_j) \in UD_{\tilde{d}}^{*}$, by the definition of minimum alternative normal form, we have that $UB_{\tilde{d}}^{k}$ is upper approximation consistent set. If one element of $UB_{\tilde{d}}^{k}$ is reduced in $UF_{\tilde{d}} = \vee_{k=1}^{p}(UB_{\tilde{d}}^{k})$, without loss of generality the result denoted by $UB_{\tilde{d}}^{k'}$, then there exist certainly $(x_{i_0}, x_{j_0}) \in UD_{\tilde{d}}^{*}$ such that $UB_{\tilde{d}}^{k'} \cap UD_{\tilde{d}}(x_{i_0}, x_{j_0}) = \varnothing$. So $UB_{\tilde{d}}^{k'}$ is not an upper approximation consistent set. So $UB_{\tilde{d}}^{k}$ is an upper approximation reduction of the ordered decision table $\mathcal{I}^{\wr\succeq}$.

On the other hand, we have known that the discernibility formula of upper approximation includes all $UD_{\tilde{d}}(x_i, x_j)$. Thus there is not other upper approximation reduction besides of $UB_{\tilde{d}}^{k}$.

The proof is completed. □

Proposition 5.6. *Let $\mathcal{I}^{\wr\succeq} = (U, AT \cup \{d\}, V, f)$ be an IFODIS. The minimal disjunctive normal form of discernibility formula of lower approximation is*

$$LF_{\tilde{d}} = \bigvee_{k=1}^{p} \left(\bigwedge_{s=1}^{q_k} \right) a'_s \,.$$

Denote $LB_{\tilde{d}}^{k} = \{a'_s \mid s = 1, 2, \cdots, q_k\}$, then $\{LB_{\tilde{d}}^{k} \mid k = 1, 2, \cdots, p\}$ is just set of all lower approximation reductions of $\mathcal{I}^{\wr}$.

Proof. It is similar to Proposition 5.5. □

Example 5.1. (Continued from Example 3.1) Compute the upper approximation reduction and lower approximation reduction of the ordered decision table with fuzzy decision in Table 1.

By computing, we can easily obtain the upper and lower approximation discernibility matrices in the following tables (See Tables 2 and 3).

Therefore, by Propositions 5.5 and 5.6, we have

$$\begin{aligned} UF_{\tilde{d}} &= (a_1 \vee a_2 \vee a_3) \wedge (a_1 \vee a_3) \wedge (a_1 \vee a_2) \wedge (a_3) \\ &= (a_1 \vee a_2) \wedge (a_3) \\ &= (a_1 \wedge a_3) \vee (a_2 \wedge a_3), \\ LF_{\tilde{d}} &= (a_1 \vee a_2 \vee a_3) \wedge (a_1 \vee a_3) \wedge (a_1 \vee a_2) \wedge (a_2) \wedge (a_3) \\ &= (a_2 \wedge a_3) \,. \end{aligned}$$

Table 2. Upper approximation discernibility matrix of the system in Example 2.1.

x_i/x_j	x_1	x_2	x_3	x_4	x_5	x_6
x_1	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
x_2	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
x_3	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
x_4	$\{a_1, a_3\}$	$\{a_3\}$	$\{a_1, a_3\}$	$\emptyset$	$\{a_3\}$	$\emptyset$
x_5	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
x_6	$\{a_1, a_3\}$	$\{a_3\}$	AT	$\{a_1, a_2\}$	$\{a_3\}$	$\emptyset$

Table 3. Lower approximation discernibility matrix of the system in Example 2.1.

x_i/x_j	x_1	x_2	x_3	x_4	x_5	x_6
x_1	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
x_2	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
x_3	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
x_4	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
x_5	AT	$\{a_2\}$	$\{a_1, a_2\}$	$\{a_1, a_2\}$	$\emptyset$	$\{a_2\}$
x_6	$\{a_1, a_3\}$	$\{a_3\}$	AT	$\{a_1, a_2\}$	$\{a_3\}$	$\emptyset$

Thus, we can conclude that $\{a_1, a_3\}$and$\{a_2, a_3\}$ are all the upper approximation reductions, and $\{a_2, a_3\}$ is all the lower approximation reduction of IFODIS, which accord with the result of Example 2.1.

In an ordered information system, an atomic expression over a single attribute a is defined as $(a, \geq)$. For any $A \subseteq AT$, an expression over A in ordered information systems is defined by $\bigwedge\limits_{a \in A} e(a)$, where $e(a)$ is an atomic expression over a. Given $a \in AT$, $v_1 \in V_a$, an atomic formula over a single attribute a is defined as $(a, \geq)$. For any $A \in AT$, a formula over A in ordered information system is denoted by $M(A)$. Let the formulas $\phi \in M(A)$, $||\phi||$ denotes the set of objects satisfying formula ϕ. For example, $(a, \geq, v_1)$, is atomic formula, then

$$||(a, \geq, v_1)|| = \{x \in U | f(x, a) \geq v_1\} .$$

However, in an intuitionistic fuzzy ordered information system, we modify the definition of a formula over a according to the dominance relation $R_A^{\wr\succeq}$ as follows

$$||(a, \succeq, v_1)|| = \{x \in U | f(x, a) \succeq v_1\} ,$$

where $f(x, a) \succeq v_1$ denotes that $\mu_a(x) \geq \mu_{v_1}(x)$ and $\nu_a(x) \leq \nu_{v_1}(x)$, $v_1 = (\mu_{v_1}, \nu_{v_1})$.

Now we consider an IFODTS $\mathcal{I}^{\wr\succeq} = (U, AT \cup \{d\}, V, f)$ and a subset of attributes $A \subseteq AT$. For formulas $\phi \in M(A)$, a decision rule, denoted by $\phi \to \varphi$, is read "if ϕ then φ." The formula ϕ is called the rule's antecedent, and the formula φ is called the rules consequent. We say that an object supports a decision rule if it matches both the condition and the decision parts of the rule. On the other hand, an object is covered by a decision rule if it matches the condition parts of the rule.

There are two types of dominance rules to be considered as follows:

(1) certain dominance rules with the following syntax:
if $(f(x,a_1) \succeq v_{a_1}) \wedge (f(x,a_2) \succeq v_{a_2}) \wedge \cdots \wedge (f(x,a_k) \succeq v_{a_k})$, then $x \in D_i$;

(2) possible dominance rules with the following syntax:
if $(f(x,a_1) \succeq v_{a_1}) \wedge (f(x,a_2) \succeq v_{a_2}) \wedge \cdots \wedge (f(x,a_k) \succeq v_{a_k})$, then x possible belong to D_i.

Now we employ an example to illustrate dominance rules extracted from intuitionistic fuzzy ordered decision information system.

Example 5.2. (Continued from Examples 3.1 and 5.1) Let us consider an IFODIS in Table 1.

We can obtain the following set of dominance rules from the Table 1:

(1) certain dominance rules with the following syntax:
$r_1 : (a_1 \succeq (0.8, 0.1)) \wedge (a_2 \succeq (0.8, 0.1)) \wedge (a_3 \succeq (0.6, 0.4)) \to (d = 3)$// supported by objects x_5;
$r_2 : (a_1 \succeq (0.8, 0.1)) \wedge (a_2 \succeq (0.6, 0.4)) \wedge (a_3 \succeq (0.8, 0.1)) \to (d = 1)$// supported by objects x_6;

(2) possible dominance rules with the following syntax:
$r_3 : (a_1 \succeq (0.2, 0.7)) \wedge (a_2 \succeq (0.6, 0.4)) \wedge (a_3 \succeq (0.5, 0.2)) \to (d = 1) \vee (d = 2) \vee (d = 3)$// supported by objects x_1;
$r_4 : (a_1 \succeq (0.8, 0.1)) \wedge (a_2 \succeq (0.6, 0.4)) \wedge (a_3 \succeq (0.6, 0.4)) \to (d = 1) \vee (d = 2) \vee (d = 3)$// supported by objects x_2;
$r_5 : (a_1 \succeq (0.2, 0.7)) \wedge (a_2 \succeq (0.1, 0.8)) \wedge (a_3 \succeq (0.6, 0.4)) \to (d = 1) \vee (d = 2) \vee (d = 3)$// supported by objects x_3;
$r_6 : (a_1 \succeq (0.3, 0.5)) \wedge (a_2 \succeq (0.1, 0.8)) \wedge (a_3 \succeq (0.8, 0.1)) \to (d = 2) \vee (d = 3)$ supported by objects x_4.

Where r_1, r_2 are certain dominance rules, r_3, r_4, r_5, r_6 are possible dominance rules.

According to Example 5.1, we know the attribute a_1 is not necessary to extract certain dominance rules, and the attribute a_3 is indispensable to extract possible dominance rules. Through a upper and lower approximation reduction, one can obtain more briefer dominance rules. For example, by taking the upper approximation reduction $\{a_1, a_3\}$ and lower approximation reduction $\{a_2, a_3\}$.The six dominance rules in Example 5.2 can be simply represented as follows.

(3) certain dominance rules with the following syntax:
$r_1 : (a_2 \succeq (0.8, 0.1)) \wedge (a_3 \succeq (0.6, 0.4)) \to (d = 3)$// supported by objects x_5;
$r_2 : (a_2 \succeq (0.6, 0.4)) \wedge (a_3 \succeq (0.8, 0.1)) \to (d = 1)$// supported by objects x_6;

(4) possible dominance rules with the following syntax:

$r_3 : (a_1 \succeq (0.2, 0.7)) \wedge (a_3 \succeq (0.5, 0.2)) \to (d = 1) \vee (d = 2) \vee (d = 3)$// supported by objects x_1;

$r_4 : (a_1 \succeq (0.8, 0.1)) \wedge (a_3 \succeq (0.6, 0.4)) \to (d = 1) \vee (d = 2) \vee (d = 3)$// supported by objects x_2;

$r_5 : (a_1 \succeq (0.2, 0.7)) \wedge (a_3 \succeq (0.6, 0.4)) \to (d = 1) \vee (d = 2) \vee (d = 3)$// supported by objects x_3;

$r_6 : (a_1 \succeq (0.3, 0.5)) \wedge (a_3 \succeq (0.8, 0.1)) \to (d = 2) \vee (d = 3)$// supported by objects x_4.

Where r_1, r_2 are certain dominance rules, r_3, r_4, r_5, r_6 are possible dominance rules.

6. Case Study

Fund has become an increasingly important source of financing for people. For an decision maker, he may need to adopt a better one from some possible fund projects or find some directions from existing successful fund projects before investing. The purpose of this section is, through a fund investment issue, to illustrate how to make a decision by using the approaches proposed in this paper.

Let us consider an fund investment issue. There are ten fund projects x_i $(i = 1, 2, \ldots, 10)$ can be considered. They can be evaluated from the view of profit factors. Profit factors are classified into five factors, which are market, technology, management, environment and production. These five factors are all increasing preference and the value of each project under each factor is given by a evaluation expert through an intuitionistic number. Table 4 is an evaluation table about fund investment given by a expert, where $U = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}\}$, AT = {Market, Technology, Management, Environment, Production} and d = {Venture}, For convenience, in the sequel, Mr, T, Mn, E, P_d and P_f will stand for Market, Technology, Management, Environment, Production, and Profit, respectively.

Table 4. An intuitionistic fuzzy ordered information system about fund investment.

U	Market	Technology	Management	Environment	Production	Profit
x_1	$(0.3, 0.5)$	$(0.6, 0.4)$	$(0.5, 0.2)$	$(0.7, 0.1)$	$(0.5, 0.4)$	Low
x_2	$(0.2, 0.7)$	$(0.1, 0.8)$	$(0.4, 0.5)$	$(0.2, 0.7)$	$(0.2, 0.8)$	Low
x_3	$(0.2, 0.7)$	$(0.1, 0.8)$	$(0.4, 0.5)$	$(0.7, 0.1)$	$(0.2, 0.8)$	Low
x_4	$(0.1, 0.8)$	$(0.1, 0.8)$	$(0.1, 0.8)$	$(0.2, 0.7)$	$(0.2, 0.8)$	Low
x_5	$(0.8, 0.1)$	$(0.8, 0.1)$	$(0.8, 0.1)$	$(0.9, 0.0)$	$(0.7, 0.1)$	High
x_6	$(0.4, 0.6)$	$(0.8, 0.1)$	$(0.6, 0.4)$	$(0.9, 0.0)$	$(0.7, 0.1)$	Low
x_7	$(0.3, 0.5)$	$(0.7, 0.3)$	$(0.5, 0.1)$	$(0.7, 0.1)$	$(0.6, 0.2)$	High
x_8	$(0.7, 0.2)$	$(0.8, 0.1)$	$(0.7, 0.1)$	$(1.0, 0.0)$	$(0.7, 0.1)$	High
x_9	$(0.7, 0.2)$	$(0.9, 0.0)$	$(0.7, 0.1)$	$(0.8, 0.2)$	$(0.9, 0.0)$	High
x_{10}	$(0.8, 0.1)$	$(0.9, 0.0)$	$(0.8, 0.1)$	$(0.6, 0.3)$	$(0.9, 0.0)$	High

From Table 4, we have that

$$U/R_{AT}^{\wr\succeq} = \{[x_1]_{AT}^{\wr\succeq}, [x_2]_{AT}^{\wr\succeq}, [x_3]_{AT}^{\wr\succeq}, [x_4]_{AT}^{\wr\succeq}, [x_5]_{AT}^{\wr\succeq}, \\ [x_6]_{AT}^{\wr\succeq}, [x_7]_{AT}^{\wr\succeq}, [x_8]_{AT}^{\wr\succeq}, [x_9]_{AT}^{\wr\succeq}, [x_{10}]_{AT}^{\wr\succeq}\},$$

where

$$\begin{aligned}
[x_1]_{AT}^{\wr\succeq} &= \{x_1, x_5, x_7, x_8\};\\
[x_2]_{AT}^{\wr\succeq} &= \{x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_9, x_{10}\};\\
[x_3]_{AT}^{\wr\succeq} &= \{x_1, x_3, x_5, x_6, x_7, x_8\};\\
[x_4]_{AT}^{\wr\succeq} &= \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}\};\\
[x_5]_{AT}^{\wr\succeq} &= \{x_5\};\\
[x_6]_{AT}^{\wr\succeq} &= \{x_5, x_6, x_8\};\\
[x_7]_{AT}^{\wr\succeq} &= \{x_5, x_7, x_8\};\\
[x_8]_{AT}^{\wr\succeq} &= \{x_8\};\\
[x_9]_{AT}^{\wr\succeq} &= \{x_9\};\\
[x_{10}]_{AT}^{\wr\succeq} &= \{x_{10}\}.
\end{aligned}$$

From the definition of dominance degree, we can get the dominance matrix of this table with respect to $U/R_{AT}^{\wr\succeq}$ as

$$M_{AT}^{\wr\succeq} = \begin{bmatrix}
1 & 1 & 1 & 1 & 0.7 & 0.8 & 0.9 & 0.7 & 0.6 & 0.6\\
0.5 & 1 & 0.7 & 1 & 0.2 & 0.4 & 0.4 & 0.2 & 0.2 & 0.2\\
0.8 & 1 & 1 & 1 & 0.5 & 0.7 & 0.7 & 0.5 & 0.4 & 0.4\\
0.4 & 0.9 & 0.6 & 1 & 0.1 & 0.3 & 0.3 & 0.1 & 0.1 & 0.1\\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0.9 & 0.9 & 0.9\\
0.9 & 1 & 1 & 1 & 0.8 & 1 & 0.9 & 0.8 & 0.7 & 0.7\\
1 & 1 & 1 & 1 & 0.8 & 0.9 & 1 & 0.8 & 0.7 & 0.7\\
1 & 1 & 1 & 1 & 0.9 & 1 & 1 & 1 & 0.9 & 0.9\\
0.9 & 1 & 0.9 & 1 & 0.9 & 0.9 & 0.9 & 0.9 & 1 & 0.9\\
0.9 & 1 & 0.9 & 1 & 0.9 & 0.9 & 0.9 & 0.9 & 0.9 & 1
\end{bmatrix}.$$

Therefore, one can obtain that

$$\begin{aligned}
&D_{AT}(x_1) = 0.83, \ D_{AT}(x_2) = 0.48, \quad D_{AT}(x_3) = 0.7, \quad D_{AT}(x_4) = 0.39,\\
&D_{AT}(x_5) = 0.97, \ D_{AT}(x_6) = 0.88, \quad D_{AT}(x_7) = 0.89, \ D_{AT}(x_8) = 0.97,\\
&D_{AT}(x_9) = 0.93, \ D_{AT}(x_{10}) = 0.93.
\end{aligned}$$

In what follows, we rank these five projects according to the number of $D_{AT}(x_i)$. A project with whole dominance degree implies that it has higher investment venture.

$$x_5 = x_8 \succcurlyeq_{AT} x_9 = x_{10} \succcurlyeq_{AT} x_7 \succcurlyeq_{AT} x_6 \succcurlyeq_{AT} x_1 \succcurlyeq_{AT} x_3 \succcurlyeq_{AT} x_2 \succcurlyeq_{AT} x_4 .$$

Thus, the investment profit of project x_5 and x_8 are highest and that of project x_4 is lowest. The decision maker may select the project x_5 and x_8 to invest.

From Table 4, it is easy to see that $\mathbf{d}= \{D_1, D_2\}$, where

$$D_1 = \{x_1, x_2, x_3, x_4, x_6, x_7\}, \quad D_2 = \{x_5, x_8, x_9, x_{10}\} .$$

From Definition 3.1, we have that

$$\underline{R_A^{\succcurlyeq}}(D_1) = \emptyset, \qquad \overline{R_A^{\succcurlyeq}}(D_1) = \{x_1, x_2, x_3, x_4, x_6, x_7\};$$
$$\underline{R_A^{\succcurlyeq}}(D_2) = \{x_5, x_8, x_9, x_{10}\}, \qquad \overline{R_A^{\succcurlyeq}}(D_2) = U.$$

Therefore, we can obtain the following set of dominance rules from the considered IFODIS:

(1) certain dominance rules with the following syntax:

$r'_1 : (Mr \succeq (0.7, 0.2)) \wedge (T \succeq (0.8, 0.1)) \wedge (Mn \succeq (0.7, 0.1)) \wedge (E \succeq (0.6, 0.3)) \wedge (P_d \succeq (0.7, 0.1)) \to (P_f = \text{High})$// supported by objects x_5, x_8, x_9, x_{10}.

(2) possible dominance rules with the following syntax:

$r'_2 : (Mr \succeq (0.3, 0.5)) \wedge (T \succeq (0.6, 0.4)) \wedge (Mn \succeq (0.5, 0.2)) \wedge (E \succeq (0.7, 0.1)) \wedge (P_d \succeq (0.5, 0.4)) \to (P_f = \text{Low})$// supported by objects x_1;

$r'_3 : (Mr \succeq (0.2, 0.7)) \wedge (T \succeq (0.1, 0.8)) \wedge (Mn \succeq (0.4, 0.5)) \wedge (E \succeq (0.2, 0.7)) \wedge (P_d \succeq (0.2, 0.8)) \to (P_f = \text{Low})$// supported by objects x_2;

$r'_4 : (Mr \succeq (0.2, 0.7)) \wedge (T \succeq (0.1, 0.8)) \wedge (Mn \succeq (0.4, 0.5)) \wedge (E \succeq (0.7, 0.1)) \wedge (P_d \succeq (0.2, 0.8)) \to (P_f = \text{Low})$// supported by objects x_3;

$r'_5 : (Mr \succeq (0.1, 0.8)) \wedge (T \succeq (0.1, 0.8)) \wedge (Mn \succeq (0.1, 0.8)) \wedge (E \succeq (0.2, 0.7)) \wedge (P_d \succeq (0.2, 0.8)) \to (P_f = \text{Low})$// supported by objects x_4;

$r'_6 : (Mr \succeq (0.4, 0.6)) \wedge (T \succeq (0.8, 0.1)) \wedge (Mn \succeq (0.6, 0.4)) \wedge (E \succeq (0.9, 0.0)) \wedge (P_d \succeq (0.7, 0.1)) \to (P_f = \text{Low})$// supported by objects x_6;

$r'_7 : (Mr \succeq (0.3, 0.5)) \wedge (T \succeq (0.7, 0.3)) \wedge (Mn \succeq (0.5, 0.1)) \wedge (E \succeq (0.7, 0.1)) \wedge (P_d \succeq (0.6, 0.2)) \to (P_f = \text{Low})$// supported by objects x_5, x_8, x_9, x_{10}.

To extract much simpler dominance rules, we compute the lower and upper approximation reductions of this decision system. The lower and upper approximation reductions of this decision system can be obtained by the proposed approximation reduction approach in Sec. 5. Tables 5 and 6 are upper and lower approximation discernibility matrix of Table 4.

From Tables 5 and 6, one can obtain that

$$\begin{aligned} LF_{\tilde{d}} &= (a_1 \vee a_2 \vee a_3 \vee a_4 \vee a_5) \wedge (a_1 \vee a_2 \vee a_3 \vee a_5) \wedge (a_1 \vee a_3 \vee a_4) \wedge (a_1 \vee a_3) \\ &= a_1 \vee a_3 \end{aligned}$$

$$\begin{aligned} UF_{\tilde{d}} &= (a_1 \vee a_2 \vee a_3 \vee a_4 \vee a_5) \wedge (a_1 \vee a_2 \vee a_3 \vee a_5) \wedge (a_1 \vee a_3 \vee a_4) \wedge (a_1 \vee a_3) \\ &= a_1 \vee a_3 . \end{aligned}$$

Table 5. Lower approximation discernibility matrix of Table 4.

x_i/x_j	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
x_1	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
x_2	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
x_3	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
x_4	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
x_5	AT	AT	AT	AT	$\emptyset$	a_1a_3	AT	$\emptyset$	$\emptyset$	$\emptyset$
x_6	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
x_7	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
x_8	AT	AT	AT	AT	$\emptyset$	$a_1a_3a_4$	AT	$\emptyset$	$\emptyset$	$\emptyset$
x_9	AT	AT	AT	AT	$\emptyset$	$a_1a_3a_4$	AT	$\emptyset$	$\emptyset$	$\emptyset$
x_{10}	$a_1a_2a_3a_5$	AT	$a_1a_2a_3a_5$	AT	$\emptyset$	$a_1a_2a_3a_5$	$a_1a_2a_3a_5$	$\emptyset$	$\emptyset$	$\emptyset$

Table 6. Upper approximation discernibility matrix of Table 4.

x_i/x_j	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
x_1	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
x_2	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
x_3	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
x_4	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
x_5	AT	AT	AT	AT	$\emptyset$	a_1a_3	AT	$\emptyset$	$\emptyset$	$\emptyset$
x_6	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
x_7	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
x_8	AT	AT	AT	AT	$\emptyset$	$a_1a_3a_4$	AT	$\emptyset$	$\emptyset$	$\emptyset$
x_9	AT	AT	AT	AT	$\emptyset$	$a_1a_3a_4$	AT	$\emptyset$	$\emptyset$	$\emptyset$
x_{10}	$a_1a_2a_3a_5$	AT	$a_1a_2a_3a_5$	AT	$\emptyset$	$a_1a_2a_3a_5$	$a_1a_2a_3a_5$	$\emptyset$	$\emptyset$	$\emptyset$

Hence, there are only one lower approximation approximation reductions in this intuitionistic fuzzy ordered decision information system about fund investment, which is {Market, Management}. Lower approximation reduction is keeps certain dominance rules invariant. Through this lower approximation reduction, one can obtain more briefer certain dominance rules as following:

(3) certain dominance rules with the following syntax:

r_1'': $(Mr \succeq (0.7, 0.2)) \wedge (Mn \succeq (0.7, 0.1)) \rightarrow (P_f = \text{High})$// supported by objects x_5, x_8, x_9, x_{10}.

There are only one upper approximation approximation reductions in this intuitionistic fuzzy ordered decision information system about fund investment, which is {Market, Management}. Upper approximation reduction is keeps possible dominance rules invariant. Through this upper approximation reduction, one can obtain more briefer possible dominance rules as following:

(4) possible dominance rules with the following syntax:

r_2'': $(Mr \succeq (0.3, 0.5)) \wedge (Mn \succeq (0.5, 0.2)) \rightarrow (P_f = \text{Low})$// supported by objects x_1;

r_3'': $(Mr \succeq (0.2, 0.7)) \wedge (Mn \succeq (0.4, 0.5)) \to (P_f = \text{Low})$// supported by objects x_2;

r_4'': $(Mr \succeq (0.2, 0.7)) \wedge (Mn \succeq (0.4, 0.5)) \to (P_f = \text{Low})$// supported by objects x_3;

r_5'': $(Mr \succeq (0.1, 0.8)) \wedge (Mn \succeq (0.1, 0.8)) \to (P_f = \text{Low})$// supported by objects x_4;

r_6'': $(Mr \succeq (0.4, 0.6)) \wedge (Mn \succeq (0.6, 0.4)) \to (P_f = \text{Low})$// supported by objects x_6;

r_7'': $(Mr \succeq (0.3, 0.5)) \wedge (Mn \succeq (0.5, 0.1)) \to (P_f = \text{Low})$// supported by objects x_7.

Where the rule r_1'' is a certain dominance rule and rules $r_2'', r_3'', r_4'', r_5'', r_6'', r_7''$ are possible dominance rules. Therefore marker and management are two important factor for this intuitionistic fuzzy issue.

7. Conclusions

Rough set theory is a new mathematical tool to deal with vagueness and uncertainty. Development of a rough computational method is one of the most important research tasks. While, in practise, intuitionistic fuzzy ordered information system confines the applications of classical rough set theory. In this article, we mainly considered some important concepts and properties in this system. We defined two approximation operators and established the rough set approach to intuitionistic fuzzy ordered information systems. For extracting dominance rules, For extracting dominance rules, we have discussed intuitionistic fuzzy ordered decision information system and dominance rules extracted from this types of decision information system. In order to extract much simpler dominance rules, based on the discernibility matrices, we have proposed approximation reduction of intuitionistic fuzzy ordered decision information system, and presented method of the reduction respectively. The approaches show how to simplify an intuitionistic fuzzy ordered decision information system and find much simpler dominance rules directly from an intuitionistic fuzzy ordered decision information system.

Acknowledgements

This work is supported by National Natural Science Foundation of China (Nos. 61105041, 71071124, 11071284 and 11001227), National Natural Science Foundation of CQ CSTC (No. cstc2011jjA40037), Science and Technology Program of Board of Education of Chongqing (KJ120805) and Graduate Innovation Foundation of Chongqing University of Technology (No. YCX2011312).

References

1. K. Atanassov, Intuitionistic fuzzy sets, *Fuzzy Sets and Systems* **20** (1986) 87–96.
2. K. Atanassov, *Intuitionistic Fuzzy Sets: Theory and Applications* (Physica-Verlag, Heidelberg, 1999).
3. K. Atanassov, Answer to D. Dubois, S. Gottwald, P. Hajek, J. Kacprzyk and H. Prade's paper, "Terminological diffulties in fuzzy set theory — the case of Intuitionistic Fuzzy Sets", *Fuzzy Sets and Systems* **156** (2005) 496–499.
4. C. Cornelis, G. Deschrijver and E. E. Kerre, Implication in intuitionistic fuzzy and interval-valued fuzzy set theory: construction, classification, application, *Int. J. Approximate Reasoning* **35** (2004) 55–95.
5. G. Cattaneo and D. Ciucci, Basic intuitionistic principles in fuzzy set theories and its extensions (a terminological debate on Atanassov IFS), *Fuzzy Sets and Systems* **157** (2006) 3198–3219.
6. K. Chakrabarty, T. Gedeon and L. Koczy, Intuitionistic fuzzy rough set, in *Proc. Fourth Joint Conf. Information Sciences* (*JCIS*), Durham, NC (1998), pp. 211–214.
7. D. Coker, Rough sets are intuitionistic L-fuzzy sets, *Fuzzy Sets and Systems* **96** (1998) 381–383.
8. C. Cornelis, M. D. Cock and E. E. Kerre, Intuitionistic fuzzy rough sets: at the crossroads of imperfect knowledge, *Expert Systems* **20** (2003) 260–270.
9. K. Dembczynski, R. Pindur and R. Susmaga, Generation of exhaustive set of rules within Dominance-based rough set approach, *Electronic Notes Theory Computer Science* **82**(4) (2003).
10. K. Dembczynski, R. Pindur and R. Susmaga, Dominance-based rough set classifier without induction of decision rules, *Electronic Notes Theory Computer Science* **82**(4) (2003).
11. G. Deschrijver and E. E. Kerre, On the position of intuitionistic fuzzy set theory in the framework of theories modelling imprecision, *Information Sciences* **177** (2007) 1860–1866.
12. D. Dubois, S. Gottwald, P. Hajek, J. Kacprzyk and H. Prade, Terminological dif?ulties in fuzzy set theory — the case of Intuitionistic Fuzzy Sets", *Fuzzy Sets and Systems* **156** (2005) 485–491.
13. S. Greco, B. Matarazzo and R. Slowinski, Rough approximation of a preference relatioin by dominance relatioin, ICS Research Report 16/96, Warsaw University of Technology; 1996 and in *Europe Journal of Operation Research* **117** (1999) 63–83.
14. S. Greco, B. Matarazzo and R. Slowinski, A new rough set approach to multicriteria and multiattribute classificatioin, in L. Polkowsik and A. Skowron (eds.), Rough sets and current trends in computing (RSCTC'98), *Lecture Notes in Artificial Intelligence*, Vol. 1424 (Springer-Verlag, Berlin, 1998), pp. 60–67.
15. S. Greco, B. Matarazzo and R. Slowinski, A new rough sets approach to evaluation of bankruptcy risk, in X. Zopounidis (ed.), *Operational Tools in the Management of Financial Risks* (Kluwer, Dordrecht, 1999), pp. 121–136.
16. S. Greco, B. Matarazzo and R. Slowinski, Rough set theory for multicriteria decision analysis, *Eur. J. Oper. Res.* **129** (2001) 11–47.
17. S. Greco, B. Matarazzo and R. Slowinski, Rough sets methodology for sorting problems in presence of multiple attributes and criteria, *Eur. J. Oper. Res.* **138** (2002) 247–259.
18. S. Greco, B. Matarazzo and R. Slowinski, Dominance-based Rough Set approach as a proper way of handling graduality in rough set theory", *Transaction on Rough Sets VII*, Lecture Notes in Computer Science, Vol. 4400, 2007, pp. 36–52.

19. S. P. Jena and S. K. Ghosh, Intuitionistic fuzzy rough sets, *Notes on Intuitionistic Fuzzy Sets* **8** (2002) 1–18.
20. Y. Leuang, W. Z. Wu and W. X. Zhang, Knowledge acquisition in incomplete information systems: a rough set approach, *Eur. J. Oper. Res.* **168**(1) (2006) 164–180.
21. J. Y. Liang and Z. Z. Shi, The information entropy, rough entropy and knowledge granulation in rough set theory, *Int. J. Uncertainty, Fuzziness and Knowledge-Based Systems* **12**(1) (2004) 37–46.
22. L. Lin, X. H. Yuan and Z. Q. Xia, Multicriteria fuzzy decision-making methods based on intuitionistic fuzzy sets, *J. Comput. Syst. Sci.* **73** (2007) 84–88.
23. S. Nanda and S. Majumda, Fuzzy rough sets, *Fuzzy Sets and Systems* **45** (1992) 157–160.
24. Z. Pawlak, Rough sets, *Int. J. Computer and Information Science* **11** (1982) 341–356.
25. Z. Pawlak and A. Skowron, Rudiments of rough sets, *Information Sciences* **177** (2007) 3–27.
26. Z. Pawlak and A. Skowron, Rough sets: some extensions, *Information Sciences* **177** (2007) 28–40.
27. Y. H. Qian, J. Y. Liang and C. Y. Dang, Interval ordered informatin systems, *Computer and Mathematics with Applications* **56** (2008) 1994–2009.
28. Y. H. Qian and J. Y. Liang, Combination entropy and combination granulation in rough set theory, *Int. J. Uncertainty, Fuzziness and Knowledge-Based Systems* **12**(1) (2004) 37–46.
29. A. M. Radzikowska, Rough approximation operations based on IF sets, in *ICAISC 2006, Lecture Notes in Computer Science*, Vol. 4029 (Springer, Berlin, 2006), pp. 528–537.
30. S. Rizvi, H. J. Naqvi and D. Nadeem, Rough intuitionistic fuzzy set, in *Proc. Sixth Joint Conf. Information Sciences* (*JCIS*), Durham, NC, 2002, pp. 101–104.
31. S. K. Samanta and T. K. Mondal, Intuitionistic fuzzy rough sets and rough intuitionistic fuzzy sets, *J. Fuzzy Mathematics* **9** (2001) 561–582.
32. M. W. Shao and W. X. Zhang, Dominance relation and relus in an incomplete ordered information system, *Int. J. Intell. Syst.* **20** (2005) 13–27.
33. A. Skowron and J. Stepaniuk, Tolerance approxximation space, *Fundamental Information* **27** (1996) 245–253.
34. R. Susmaga, R. Slowinski, S. Greco and B. Matarazzo, Generation of reducts and rules in multi-attribute and multi-criteria classification, *Control and Cybernetics* **4** (2000) 969–988.
35. K. Thangavel and A. Pethalakshmi, Dimensionality reduction based on rough set theory: A review, *Applied Soft Computing* **9** (2009) 1–12.
36. L. K. Vlachos and G. D. Sergiadis, Intuitionistic fuzzy information — applications to pattern recognition, *Pattern Recognition Letters* **28** (2007) 197–206.
37. W. Z. Wu, Y. Leung and W. X. Zhang, Connections between rough set theory and Dempster-Shafer theory of evidence, *Int. J. General Systems* **31** (2002) 405–430.
38. W. Z. Wu, M. Zhang, H. Z. Li and J. S. Mi, Knowledge reduction in random information systems via Dempster-Shafer theory of evidence, *Information Sciences* **174** (2005) 143–164.
39. W. H. Xu, Y. Li and X. W. Liao, Approaches to attribute reductions based on rough set and matrix computation in inconsistent ordered information systems, *Knowledge-Based Systems* **27** (2012) 78–91.
40. W. H. Xu and W. X. Zhang, Measuring roughness of generalized rough sets induced by a covering, *Fuzzy Sets and Systems* **158** (2007) 2443–2455.

41. W. H. Xu, X. Y. Zhang, J. M. Zhong and W. X. Zhang, Attribute reduction in ordered information systems based on evidence theory, *Knowledge and Information Systems* (2010) 169–184.
42. W. H. Xu, X. Y. Zhang and W. X. Zhang, Knowledge granulation, knowledge entropy and knowledge uncertainty measurein ordered information systems, *Applied Soft Computing* **9** (2009) 1244–1251.
43. Z. S. Xu, Intuitionistic preference relations and their application in group decision making, *Information Sciences* **177** (2007) 2363–2379.
44. Y. Y. Yao, Relational interpretations of neighborhood operators and rough set approximation operators, *Information, Sciences* **101** (1998) 239–259.
45. Y. Y. Yao and P. Lingras, Interpretations of belief functions in the theory of rough sets, *Information Sciences* **104** (1998) 81–106.
46. D. R. Yu, Q. H. Hu and C. X. Wu, Uncertainty measures for fuzzy relations and their applications, *Applied Soft Computing* **7** (2007) 1135–1143.
47. L. A. Zadeh, Fuzzy sets, *Information and Control* **8** (1965) 338–353.
48. X. Y. Zhang and W. H. Xu, Rough set approach to approximation reduction in ordered decision table with fuzzy decision, *Mathematical Problems in Engineering*, Vol. 2011, Article ID 268929, 16 pages, doi:10.1155,2011.268929.
49. X. Y. Zhang and W. H. Xu, Ranking for objects and attribute reductions in intuitionistic fuzzy ordered information systems, *Mathematical Problems in Engineering*, Vol. 2012, Article ID 426901, 19 pages doi:10.1155,2012.426901.